ON THE STABILITY OF MOTION OF GYROSTATS

(OB USTOICHIVOSTI DVIZHENIIA GIROSTATOV)

PMM Vol.25, No.1, 1961, pp. 9-16

V. V. RUMIANTSEV
(Moscow)

(Received November 14, 1960)

A gyrostat [1] is a mechanical system S which consists of a solid body S_1 and other bodies S_2 which are connected to it. These other bodies are either variable or solid, but their motion relative to the body S_1 does not alter the geometry of the mass system S.

Examples of such systems are: a solid body to which there are connected axes of several (or of one) symmetric gyroscopes; or a solid body with a cavity of arbitrary shape entirely filled with a homogeneous liquid; and similar systems.

It is obvious that for a given distribution of masses in a gyrostat no change can occur in the position of the center of gravity of the principal axes and of the moments of inertia of the gyrostat with respect to any point of the solid body S_1 as the result of the internal motion of the bodies S_2.

In the present work there is investigated, by the use of the second method of Liapunov, the stability of certain motions of heavy gyrostats with one fixed point.

1. Let us suppose that the solid body S_1 has one fixed point O which we take as the origin of two rectangular coordinate systems: a fixed system $O\xi\eta\zeta$ with the axis $O\zeta$ directed upward, and a moving system $Oxyz$ whose axes coincide with the principal axes of inertia of the gyrostat S for the fixed point O.

By the theorem on the addition of velocities, the velocity vector of any point of S_2 relative to the $O\xi\eta\zeta$-coordinate system is equal to the geometric sum of the transfer velocity of this point (in its motion with S_1) and its relative velocity (in its motion with respect to S_1). The vector of the moment of the entire motion of the S_2 body can be represented as the geometric sum of the vectors of the moment of the transfer motion and the moment of the relative motion of this body. In view of what has been said, the moment of the entire motion of the gyroscope relative to the point O can be represented as the geometric sum $\mathbf{K} + \mathbf{k}$, where $\mathbf{K}$ is the moment of the motion of the entire system S considered as one solid body, and $\mathbf{k}$ is the moment of the relative motion of the body S_2. The projections of the vector $\mathbf{k}$ on the x, y, z-axes will be denoted by k_1, k_2 and k_3, while the projections of the vector $\mathbf{K}$ upon the same axes are given, respectively, by

$$K_1 = Ap, \qquad K_2 = Bq, \qquad K_3 = Cr$$

where A, B and C are the principal moments of inertia of the gyrostat S for the point O, and p, q and r are the projections on the moving axes of the vector $\boldsymbol{\omega}$, the instantaneous velocity of the body S_1.

By the theorem on the moment of momentum we obtain the following equations for the motion of a heavy gyrostat with one fixed point:

$$\begin{aligned} A\,\frac{dp}{dt} + \frac{dk_1}{dt} + (C-B)\,qr + qk_3 - rk_2 &= P\,(z_0\gamma_2 - y_0\gamma_3) \\ B\,\frac{dq}{dt} + \frac{dk_2}{dt} + (A-C)\,rp + rk_1 - pk_3 &= P\,(x_0\gamma_3 - z_0\gamma_1) \\ C\,\frac{dr}{dt} + \frac{dk_3}{dt} + (B-A)\,pq + pk_2 - qk_1 &= P\,(y_0\gamma_1 - x_0\gamma_2) \end{aligned} \tag{1.1}$$

Here P denotes the weight of the gyrostat; the constants x_0, y_0 and z_0 are the coordinates of its center of gravity; γ_1, γ_2 and γ_3 are the cosines of the angles between the vertical axis $O\zeta$ and the moving axes x, y and z which satisfy Poisson's equations

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3, \qquad \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1, \qquad \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2 \tag{1.2}$$

Equations (1.1) and (1.2) do not, in general, suffice for the complete analysis of the motion of a heavy gyrostat with one fixed point. In addition one has to have equations of the relative motion of the body S_2 which can have different forms depending on the form of the body S_2, on the nature of the imposed connection and on the acting forces inside the system S. For example, if the body S_2 is a homogeneous liquid filling a cavity of the solid body S_1, then the equations of the relative motion can be written in the form of the hydrodynamic equations of Euler or of the Navier-Stokes equations, and of the equations of incompressibility, together with the boundary conditions on the walls of the cavity [6]. If the body S_2 represents a symmetrical rotor with an axis that is fixed relative to S_1, then the equation of the relative motion will have the form of the equation of motion of a solid body with a fixed axis, and so on.

Equations (1.1) and (1.2) will suffice for the study of the motion of a gyrostat in the case where the vector **k** is known at the start, i.e. in the case where the $k_i (i = 1, 2, 3)$ are given functions of time, or, in particular, if they are constants. For example, the k_i = const in the case of nonturbulent motion of an ideal liquid filling completely a multiply-connected cavity of S_1.

It is possible to give some first integrals of the motion of a gyrostat. Let us assume that the internal forces acting on the body S_2 have a force (potential) function U and that the connections are stationary. Then on the basis of the theorem on the kinetic energy, one can obtain the following integral of the kinetic energy:

$$\begin{gathered} Ap^2 + Bq^2 + Cr^2 + 2(pk_1 + qk_2 + rk_3) + \\ + 2(T_2 - U) + 2P(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = \text{const} \end{gathered} \tag{1.3}$$

where T_2 denotes the kinetic energy of the body S_2 in its relative motion.

If k_i = const $(i = 1, 2, 3)$, then the integral on the kinetic energy can be obtained by means of Equations (1.1) and (1.2). Indeed, let us multiply Equations (1.1) by p, q and r, respectively, and add the result. Then, in view of (1.2), we obtain the first integral

$$Ap^2 + Bq^2 + Cr^2 + 2P(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = \text{const} \tag{1.4}$$

which has the same form as it would have if the gyrostat S had been a solid body.

Let us multiply (1.1) by γ_1, γ_2 and γ_3, respectively, and add the result. Then, by (1.2), we obtain the integral of the planes

$$(Ap + k_1)\gamma_1 + (Bq + k_2)\gamma_2 + (Cr + k_3)\gamma_3 = \text{const} \tag{1.5}$$

In the case where the motion of the gyrostat is by inertia, when $x_0 = y_0 = z_0 = 0$, one can also obtain, in addition to the integrals of the form (1.5), integrals of the constancy of the moment of momentum of the system. With this in mind, let us multiply Equation (1.1), whose right sides are now zero, by $A_p + k_1$, $B_q + k_2$, and $C_r + k_3$, respectively, and add the results. After this we can easily obtain the integral

$$(Ap + k_1)^2 + (Bq + k_2)^2 + (Cr + k_3)^2 = \text{const} \tag{1.6}$$

Concurrently we note that in [1, p. 223] it is mistakenly stated that the integral of the constancy of the moment of momentum, when K_i = const, has the form

$$A^2p^2 + B^2q^2 + C^2r^2 = \text{const} \tag{1.7}$$

One can easily reveal the mistake by taking the derivative with respect to time of the left-hand side of Equation (1.7). In view of (1.1), with $x_0 = y_0 = z_0 = 0$, the obtained derivative will, in general, not vanish.

Equations (1.2) obviously admit the geometric integral

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{1.8}$$

2. Let us examine the stability of the permanent rotations of the gyrostat which is moving under inertia ($x_0 = y_0 = z_0 = 0$), in the case when the $k_i (i = 1, 2, 3)$ are given constants.

It should be noted that Zhukovskii [2] has given a geometric interpretation of the motion of a gyrostat for this case. A detailed investigation of the permanent rotations and their stability for a gyrostat moving under inertia was made by Volterra [3]. For the investigation of the stability we shall make use of the direct method of Liapunov.

Suppose the permanent axis has a fixed direction in the body, which is given by its direction cosines α, β and γ in the moving coordinates. Then the projections of the angular velocity of the S_1 body upon the moving axes will be

$$p_0 = \omega\alpha, \qquad q_0 = \omega\beta, \qquad r_0 = \omega\gamma \qquad (\omega = \text{const}) \tag{2.1}$$

Hereby, Equations (1.1) will take on the form

$$\begin{aligned}
&(C - B)\,\beta\gamma\omega^2 + \omega\,(\beta k_3 - \gamma k_2) = 0\\
&(A - C)\,\gamma\alpha\omega^2 + \omega\,(\gamma k_1 - \alpha k_3) = 0\\
&(B - A)\,\alpha\beta\omega^2 + \omega\,(\alpha k_2 - \beta k_1) = 0
\end{aligned} \tag{2.2}$$

and they will serve for the determination of the corresponding value of the angular velocity ω. Multiplying these equations by k_1, k_2 and k_3, respectively, and adding them, we obtain after cancelling out ω^2

$$(C - B)\,\beta\gamma k_1 + (A - C)\,\alpha\gamma k_2 + (B - A)\,\alpha\beta k_3 = 0 \tag{2.3}$$

In terms of the variables α, β and γ, this is the equation of a cone of the second order [1] with its vertex at the fixed point O. Equation (2.3) coincides with the cone of Staude-Mlodzeevski if one replaces k_i by the coordinates of the center of gravity x_0, y_0, z_0 of the heavy solid body [5].

The investigation of the stability of the permanent rotations of the gyrostat can be accomplished by means of the construction of a Liapunov function analogous to the one for the case of a single solid body [4,5]. Setting in the perturbed motion

$$p = p_0 + \xi_1, \qquad q = q_0 + \xi_2, \qquad r = r_0 + \xi_3$$

one can easily see that the equations of the perturbed motion admit the following first integrals:

$$\begin{aligned} V_1 &= A(\xi_1^2 + 2p_0\xi_1) + B(\xi_2^2 + 2q_0\xi_2) + C(\xi_3^2 + 2r_0\xi_3) = \text{const} \\ V_2 &= A^2(\xi_1^2 + 2p_0\xi_1) + B^2(\xi_2^2 + 2q_0\xi_2) + C^2(\xi_3^2 + 2r_0\xi_3) + \\ &\quad + 2(Ak_1\xi_1 + Bk_2\xi_2 + Ck_3\xi_3) = \text{const.} \end{aligned} \tag{2.4}$$

The Liapunov function can, for example, be constructed in the form

$$V = \lambda V_1 - V_2 = \frac{Ak_1}{p_0}\xi_1^2 + \frac{Bk_2}{q_0}\xi_2^2 + \frac{Ck_3}{r_0}\xi_3^2 \tag{2.5}$$

where, in view of Equation (2.2)

$$\lambda = \frac{Ap_0 + k_1}{p_0} = \frac{Bq_0 + k_2}{q_0} = \frac{Cr_0 + k_3}{r_0}$$

Obviously, the function (2.5) has one definite sign if the ratios k_1/p_0, k_2/q_0, k_3/r_0 have the same sign, which establishes the stability of the permanent rotations under these conditions.

Of greater interest are permanent rotations of a gyrostat with an arbitrary angular velocity ω around its principal central axes of inertia, which are possible under the condition of collinearity of the vector $\mathbf{k}$ with the permanent axis of rotation.

Suppose, for example, that $k_1 = k_2 = 0$, $k_3 = k = \text{const}$. Then Equation (2.2) admits the solution

$$\alpha = \beta = 0, \qquad \gamma = 1 \qquad (p_0 = q_0 = 0,\ r_0 = \omega) \tag{2.6}$$

for an arbitrary value of ω. Let us consider the function

$$\begin{aligned} V &= V_2 - \left(C + \frac{k}{\omega}\right)V_1 + \frac{1}{4\omega^2}\mu V_1^2 = \qquad \left(C_1 = C + \frac{k}{\omega}\right) \\ &= A(A - C_1)\xi_1^2 + B(B - C_1)\xi_2^2 + C\left(C\mu - \frac{k}{\omega}\right)\xi_3^2 + \cdots \end{aligned} \tag{2.7}$$

Here the dots stand for omitted terms of the third and fourth degree in ξ_1, ξ_2 and ξ_3. It is obvious that if

$$A \gtrless C_1, \qquad B \gtrless C_1 \tag{2.8}$$

where in both inequalities one uses simultaneously either only the upper signs or only the lower signs, then one can always choose such a value μ = const that the function (2.7) will be of definite sign. On the basis of Liapunov's theorem, the motion (2.6) will be stable under the conditions (2.8).

Next, let us consider the function

$$W = \xi_1\xi_2 \tag{2.9}$$

and its time derivative taken with the aid of the following equations of the perturbed motion:

$$A\frac{d\xi_1}{dt} + (C - B)\,\xi_2\,(\omega + \xi_3) + \xi_2 k = 0$$

$$B\frac{d\xi_2}{dt} + (A - C)\,\xi_1\,(\omega + \xi_3) - \xi_1 k = 0$$

We thus obtain

$$W' = \left(\frac{C_1 - A}{B}\xi_1^2 + \frac{B - C_1}{A}\xi_2^2\right)\omega + \left(\frac{C - A}{B}\xi_1^2 + \frac{B - C}{A}\xi_2^2\right)\xi_3 \tag{2.10}$$

Let us suppose that during the entire time of motion, the variable ξ_3 preserves the order of smallness of the quantities ξ_1 and ξ_2. In the opposite case we would have instability with respect to this variable.

Then, if the inequalities

$$C_1 \gtrless A, \qquad B \gtrless C_1 \tag{2.11}$$

hold simultaneously with both upper signs or with both lower signs, the function W' will be of definite sign in the variables ξ_1 and ξ_2. On the basis of a theorem of Chetaev [4] we can conclude that the unperturbed motion (2.6) is not stable in this case.

The quantity C_1 has the dimensions of a moment of inertia, and for the given angular velocity ω it can be considered as a "fictitious" moment of inertia of the gyrostat with respect to the permanent axis z if $C_1 > 0$. By introducing into our consideration the ellipsoid

$$Ax^2 + By^2 + C_1z^2 = 1 \tag{2.12}$$

We can formulate the results obtained, obviously, in the form of a known theorem [4] on the stability of the permanent rotations of a solid body with the ellipsoid of inertia (2.12).

One should, however, keep in mind that in the case of opposite signs of the quantities k and ω, the quantity C_1 can be non-positive. Then the condition (2.8) will be satisfied with the upper sign, and, hence, the undisturbed motion (2.6) will be stable.

Thus [3] if the ratio k/ω lies within the bounds $(A - C)$ and $(B - C)$, the corresponding permanent motion (2.6) is unstable; in the opposite case it will be stable relative to the variables p, q and r.

Hereby it is easy to establish the stability of the undisturbed motion with respect to perturbations of the quantities k_i = const [3] if it is stable with respect to the quantities p, q and r.

3. Let us consider the case when

$$A \geqslant B, \qquad x_0 = y_0 = 0, \qquad z_0 \neq 0, \qquad k_1 = k_2 = 0, \qquad k_3 = k = \text{const} \quad (3.1)$$

Equations (1.1) and (1.2) admit a particular solution

$$p = q = 0, \qquad r = \omega, \qquad \gamma_1 = \gamma_2 = 0, \qquad \gamma_3 = 1 \qquad (3.2)$$

which describes a uniform rotation of a gyrostat with an arbitrary angular velocity ω around the vertical axis z. We shall take this motion for the undisturbed motion and shall investigate its stability by setting in the disturbed motion

$$r = \omega + \xi, \qquad \gamma_3 = 1 + \zeta$$

We shall use the previous notation for the remaining variables.

The equations of the disturbed motion admit the first integrals

$$\begin{aligned}
V_1 &= Ap^2 + Bq^2 + C(\xi^2 + 2\omega\xi) + 2Pz_0\zeta = \text{const} \\
V_2 &= Ap\gamma_1 + Bq\gamma_2 + C\xi + C(\omega + \xi)\zeta + k\zeta = \text{const} \\
V_3 &= \gamma_1^2 + \gamma_2^2 + \zeta^2 + 2\zeta = 0
\end{aligned} \qquad (3.3)$$

Let us construct the functions

$$\begin{aligned}
V &= V_1 - 2\omega V_2 + (C\omega^2 + k\omega - Pz_0)V_3 + \tfrac{1}{4}\mu V_3^2 = \\
&= Ap^2 - 2A\omega p\gamma_1 + (C\omega^2 + k\omega - Pz_0)\gamma_1^2 + \\
&+ Bq^2 - 2B\omega q\gamma_2 + (C\omega^2 + k\omega - Pz_0)\gamma_2^2 + \\
&+ C\xi^2 - 2C\omega\xi\zeta + (C\omega^2 + k\omega - Pz_0 + \mu)\zeta^2 + \cdots
\end{aligned} \qquad (3.4)$$

where the repeated dots denote infinitesimals of the third or fourth order; the constant $\mu > Pz_0 - k\omega$.

According to Sylvester's criterion the condition for the positive-definiteness of the function (3.4) is the inequality

$$(C-A)\omega^2+k\omega-Pz_0>0 \tag{3.5}$$

When this condition (3.5) holds, the undisturbed motion (3.2) of an asymmetrical gyrostat will be stable with respect to the variables p, q, r, γ_1, γ_2, γ_3.

4. Let us now proceed to the consideration of the stability of the rotation of a symmetric gyrostat when the following conditions hold:

$$A=B, \qquad x_0=y_0=0, \qquad z_0\neq 0, \qquad k_1=k_2=0 \tag{4.1}$$

and the projection of the vector $\mathbf{k}$ on the z-axis is some bounded function of the time $k_3 = k(t)$ determined by the equation of the relative motion of the body S_2.

Equations (1.1) take the following form in this case:

$$\begin{aligned} A\frac{dp}{dt}+(C-A)\,qr+qk(t)&=Pz_0\gamma_2\\ A\frac{dq}{dt}+(A-C)\,rp-pk(t)&=-Pz_0\gamma_1\\ C\frac{dr}{dt}+\frac{dk(t)}{dt}&=0\end{aligned} \tag{4.2}$$

From the third of these equations we obtain the integral

$$Cr+k(t)=\mathrm{const} \tag{4.3}$$

Multiplying the first of Equations (4.2) by p and the second by q, and adding the result we obtain, in view of Equation (1.2), the next first integral

$$A(p^2+q^2)+2Pz_0\gamma_3=\mathrm{const} \tag{4.4}$$

The equations of motion (4.2) and (1.2) admit the following particular solution

$$p=q=0, \qquad \gamma_1=\gamma_2=0, \qquad Cr+k(t)=K, \qquad \gamma_3=1 \tag{4.5}$$

which describes the rotation of a gyrostat with a variable angular velocity

$$r=\frac{K}{C}-\frac{1}{C}k(t)$$

around the vertical z-axis. We take this motion as the undisturbed motion, and shall investigate its stability by setting in the perturbed motion

$$Cr + k(t) = K + \xi; \qquad \gamma_3 = 1 + \zeta \tag{4.6}$$

and retaining the previous notation for the remaining variables.

The equations of the perturbed motion, which can easily be derived with the aid of (4.6), admit the following first integrals:

$$\begin{aligned} V_1 &= A(p^2 + q^2) + 2Pz_0\zeta = \text{const} \\ V_2 &= A(p\gamma_1 + q\gamma_2) + K\zeta + \xi + \xi\zeta = \text{const} \\ V_3 &= \gamma_1^2 + \gamma_2^2 + \zeta^2 + 2\zeta = 0 \\ V_4 &= \xi = \text{const} \end{aligned} \tag{4.7}$$

Let us construct the function

$$\begin{aligned} V &= V_1 + 2\lambda V_2 - (Pz_0 + K\lambda) V_3 - 2\lambda V_4 + \frac{1}{A} V_4^2 \\ &= Ap^2 + 2A\lambda p\gamma_1 - (Pz_0 + K\lambda)\gamma_1^2 + Aq^2 + 2A\lambda q\gamma_2 - (Pz_0 + K\lambda)\gamma_2^2 + \\ &+ \frac{1}{A}\xi^2 + 2\lambda\xi\zeta - (Pz_0 + K\lambda)\zeta^2 \end{aligned} \tag{4.8}$$

In accordance with Sylvester's criterion the condition for the positive-definiteness of the function (4.8) will be the inequality

$$A\lambda^2 + K\lambda + Pz_0 < 0$$

which can be satisfied by the appropriate choice of the constant λ if

$$K^2 - 4APz_0 > 0 \tag{4.9}$$

This is a generalization of the known Maievskii condition [4].

When the condition (4.9) holds, the unperturbed motion (4.5) will be stable with respect to the quantities p, q, $Cr + k(t)$, γ_1, γ_2 and γ_3.

In the case where $k(t)$ is a given continuous bounded function we will also have stability with respect to r in view of the existence of the integral (4.3).

It is easy to see that the condition (4.9) is also necessary for the stability of the undisturbed motion (4.5). Indeed, let us consider the function

$$W = p\gamma_2 - q\gamma_1$$

and its time derivative, evaluated on the basis of the equation of the disturbed motion

$$W' = (p^2 + q^2)(1 + \zeta) - \frac{K + \xi}{A}(p\gamma_1 + q\gamma_2) + \frac{Pz_0}{A}(\gamma_1^2 + \gamma_2^2)$$

According to Sylvester's criterion the function W' will be positive-definite with respect to the variables p, q, γ_1 and γ_2 if the next inequality holds:

$$K^2 - 4APz_0 < 0 \tag{4.10}$$

Here it is assumed that the variable ζ always preserves the order of smallness of the variables p and q; in the opposite case we would have obvious instability of the undisturbed motion (4.5) with respect to γ_3. Hence, if condition (4.10) holds, the undisturbed motion is unstable, since the function W would then fulfill the conditions of Chetaev's theorem on instability.

Thus the inequality (4.9) is a necessary and sufficient condition for the stability of the undisturbed motion (4.5).

In the case where k_3 = const the integral (4.3) takes the form

$$r = \text{const}$$

and in place of the particular solution (4.5) we will have the solution (3.2).

In this case the condition for stability (4.9) becomes

$$(C\omega + k_3)^2 - 4APz_0 > 0$$

This inequality can be fulfilled by a proper selection of the quantity k_3 = const, and in the case where $\omega = 0$, i.e. the unstable equilibrium of a heavy gyrostat can be stabilized by a rotation of the body S_2.

5. We note that the results obtained above on the stability of permanent rotations of a gyrostat for the case when k_i = const (i = 1, 2, 3) are applicable, in particular, to a solid body with multiply-connected cavities which are completely filled with an ideal homogeneous liquid in a state of irrotational motion.

Zhukovskii [2] has shown that the equations of motion of such a gyrostat have the form of Equations (1.1) where A, B and C now denote the principal moments of inertia of the transformed solid body (obtained by connecting to S_1 solid bodies to replace the liquid masses), and the k_i = const are the sums of the projections of the moments of the non-turbulent motions of the liquids in the multiply-connected cavities of the solid body. The latter are expressed by means of linear functions of the principal circulations. For example, in the case of a ring-shaped cavity of rotation [2] around the z-axis

$$k_1 = k_2 = 0, \quad k_3 = \frac{m\chi}{2\pi}$$

where m is the mass of the liquid, χ is the circulation of the velocity determined by the initial motion of the liquid.

If at the initial moment when the body is at rest the liquid is at rest too, then all the k_i are zero and we have the case of one transformed body [1]. For the case of vortex motion of a liquid within the cavity of a solid body, the stability of the rotation of the gyrostat about the vertical axis is investigated in [6].

BIBLIOGRAPHY

1. Levi-Civita and Amaldi, *Kurs teoreticheskoi mekhaniki (Course in Theoretical Mechanics)*, Vol. 2, Part 2. IIL, 1951. (Russian translation).

2. Zhukovskii, N.E., O dvizhenii tverdogo tela, imeiushchego polosti, napolnennye odnorodnoi kapel'noi zhidkost'iu (On the motion of a solid body with a cavity filled with a homogeneous drop liquid). *Collected Works*, Vol. 2. Gostekhteoretizdat, 1948.

3. Volterra, V., Sur la théorie des variations des latitudes. *Acta math.* Vol. 22, 1899.

4. Chetaev, N.G., *Ustoichivost' dvizheniia (Stability of Motion)*. Gostekhteoretizdat, 1955.

5. Rumiantsev, V.V., Ustoichivost' permanentnykh vrashchenii tiazhelogo tverdogo tela (Stability of permanent rotations of a heavy solid body). *PMM* Vol. 20, No. 1, 1956.

6. Rumiantsev, V.V., Ob ustoichivosti vrashcheniia volchka s polost'iu, zapolnennoi viazkoi zhidkost'iu (On the stability of the rotation of a top with a cavity filled with a viscous liquid). *PMM* Vol. 24, No. 4, 1960.

Translated by H.P.T.